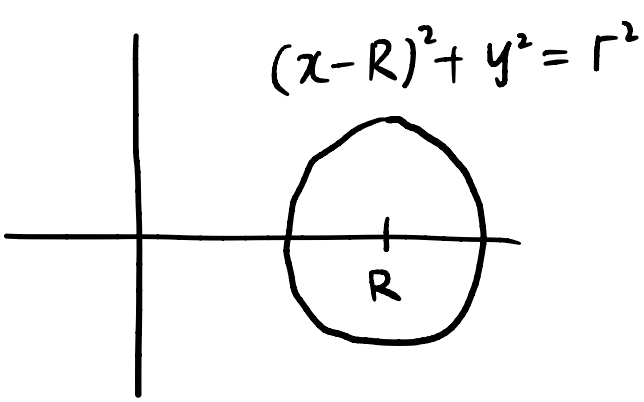
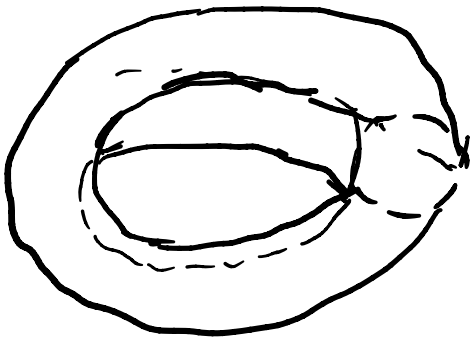
The circle with center $(R, 0)$ and radius $r < R$ is rotated around the y-axis. What is the surface area of the resulting solid?





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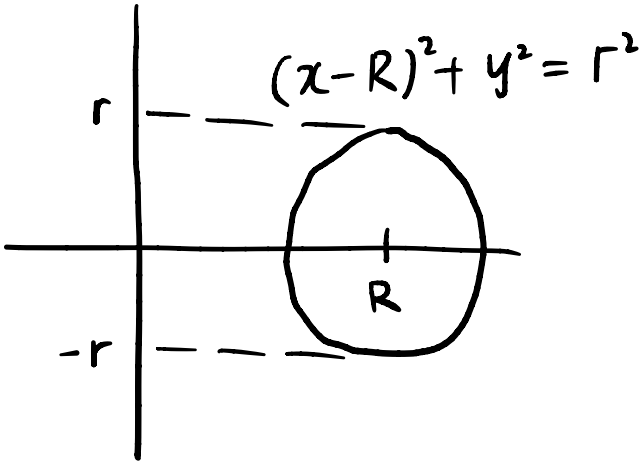

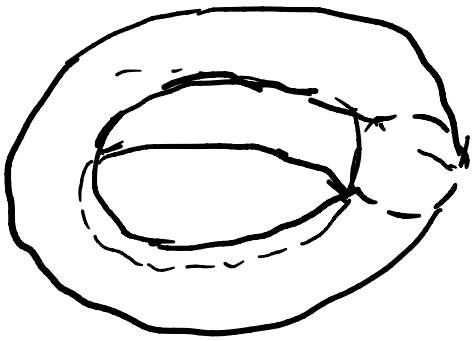

$(x-R)^2 = r^2 - y^2$

$x - R = \pm\sqrt{r^2 - y^2}$

bigger $x \Rightarrow x = R + \sqrt{r^2 - y^2}$

$dS_1 = 2\pi x\, ds$

$\Rightarrow S_1 = 2\pi \int_{-r}^{r} \left(R + \sqrt{r^2 - y^2}\right)\left(\frac{r}{\sqrt{r^2 - y^2}}\, dy\right)$

$= 2\pi \int_{-r}^{r} \frac{Rr}{\sqrt{r^2 - y^2}} + r\, dy$

$= 2\pi(Rr\pi) + 2\pi r(y)\Big|_{-r}^{r}$

$= 2\pi r(R\pi + 2r)$

$ds = \sqrt{dx^2 + dy^2}$

$\Rightarrow ds = \sqrt{x'^2 + 1}\, dy$

$x' = \frac{1}{2} \frac{1}{\sqrt{r^2 - y^2}} (-2y)$

$= -\frac{y}{\sqrt{r^2 - y^2}}$

$x'^2 = \frac{y^2}{r^2 - y^2}$

$\Rightarrow ds = \sqrt{x'^2 + 1}\, dy$

$= \sqrt{\frac{y^2 + (r^2 - y^2)}{r^2 - y^2}}\, dy$

$= \frac{r}{\sqrt{r^2 - y^2}}\, dy$

$Rr \int_{-r}^{r} \frac{1}{\sqrt{r^2 - y^2}}\, dy$

$y = r\sin u \Rightarrow dy = r\cos u\, du$

$r = r\sin u \quad y = -r$

$u = \sin^{-1} 1 \Rightarrow u = \sin^{-1} -1$

$= \frac{\pi}{2} \qquad = -\frac{\pi}{2}$

$= \frac{Rr}{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r\cos u}{\cos u}\, du$

$= Rr(u)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= Rr\pi$

$$S_2 = 2\pi \int_{-r}^{r} \left(R - \sqrt{r^2 - y^2}\right)\left(\frac{r}{\sqrt{r^2 - y^2}}\right) dy$$

$$= 2\pi r\left(R\pi - 2r\right)$$

Total Surface Area

$= S_1 + S_2$

$= 2\pi r\left(R\pi + 2r\right) + 2\pi r\left(R\pi - 2r\right)$

$= 4\pi r \cdot R\pi$

$= 4\pi^2 rR$